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# THE THREE-REGGEON VERTEX

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Abstract: Properties of the three-Reggeon coupling function are examined. The expected form is given of its dependence on variables related to generalisations of the Toller angle. It is found that the Veneziano-type models for multiparticle amplitudes contain three-Reggeon vertices, with the correct structure; this provides a non-trivial check on these models. This structure is also checked in a model based on Gribov's hybrid perturbation theory approach.

## 1. INTRODUCTION

In a previous paper [1], here referred to as I, we have examined the function $f_{\alpha_1\alpha_2}$ that couples a pair of Reggeons to a spin-zero particle. This function is defined through the high-energy behaviour of a two-particle → three-particle production process. Apart from complications introduced by the effects of signature (which were studied in a second paper [2]), it was shown that $f_{\alpha_1\alpha_2}$ takes the general form

$$f_{\alpha_1\alpha_2} = \int_{-\infty}^{\infty} \mathrm{d}y\ f(y)\Phi(\alpha_1, \alpha_2; y/\eta)\,, \tag{1.1}$$

where

$$\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Phi(\alpha_1,\alpha_2;z) = \int_0^{\infty} \mathrm{d}x_1\,\mathrm{d}x_2\, x_1^{-\alpha_1-1}\, x_2^{-\alpha_2-1}\, \mathrm{e}^{i[x_1x_2z+x_1+x_2]}\,. \tag{1.2}$$

The variable $\eta$ is linearly related to the cosine of the Toller angle [3] and is defined in eq. (2.17) below.

In this paper we study the function $F_{\alpha_1\alpha_2\alpha_3}$ that couples together three Reggeons. This function is defined through the high-energy behaviour of a three-particle → three-particle amplitude, which is not at present of great practical interest. But the properties of $F_{\alpha_1\alpha_2\alpha_3}$ are of some theoretical interest, for several reasons.

First, by putting $\alpha_3$ equal to a positive integer $p$ in $F_{\alpha_1\alpha_2\alpha_3}$, we may obtain information about the structure of the coupling of two Reggeons to a physical particle of spin $p$. Secondly, it is our hope that at some stage it will be useful to consider generalised field operators associated with Reggeons and an advance knowledge of the properties of functions such as $F_{\alpha_1\alpha_2\alpha_3}$ will be an important test of the correctness of the formalism. (An attempt to introduce such operators was made originally by Cabibbo, Horwitz and Ne'eman [4], but their operators did not have the correct Lorentz tensor properties for them obviously to be associated with Reggeons of general spin.) Thirdly, $F_{\alpha_1\alpha_2\alpha_3}$ does appear when unphysical limits are taken in production amplitudes that are of present experimental interest, for example in the two-particle → four-particle amplitude obtained by crossing from the three-particle → three-particle one. Thus a knowledge of the properties of $F_{\alpha_1\alpha_2\alpha_3}$ is necessary if we are ultimately to have a complete understanding of the properties of such amplitudes. In particular, our work provides a non-trivial check on the generalisation of the Veneziano model to multiparticle amplitudes [5, 6]. The generalisation was designed to give the amplitudes a structure in which pairs of Reggeons are coupled to particles and in I we pointed out that the coupling function takes a form that agrees with the general expression (1.1). Here we find that it happens that the model also includes three-Reggeon couplings, with a coupling function in agreement with the expected general form.

This general form is determined in sect. 2, which begins with a description of the particular asymptotic limit that must be applied to the six-point function in order to reveal the three-Reggeon coupling. Here, and throughout this paper, we omit complications resulting from signature; these may be deduced from a straight generalisation of our work in ref. [2]. In sect. 3 we consider the three-Reggeon coupling function in a particular model, based on Gribov's hybrid perturbation theory technique [7], and in sect. 4 we examine the generalised Veneziano model.

It is perhaps worth pointing out that Gribov's original paper also discussed a three-Reggeon coupling function $r_{\alpha_1\alpha_2\alpha_3}$. However, $r_{\alpha_1\alpha_2\alpha_3}$ differs from $F_{\alpha_1\alpha_2\alpha_3}$ in that it involves an integral of $F_{\alpha_1\alpha_2\alpha_3}$ with respect to the momentum transfer between a pair of the Reggeons; thus the remaining Reggeon plays a distinguished role. We believe that our representation for $F_{\alpha_1\alpha_2\alpha_3}$ given in sect. 2 and extended to incorporate signature effects, can be used to obtain information about the structure of $r_{\alpha_1\alpha_2\alpha_3}$. But the simple models of sects. 3 and 4 given zero contribution to $r_{\alpha_1\alpha_2\alpha_3}$ when the appropriate integration is performed.

## 2. GENERAL THEORY

Consider a six-point function describing the process

$$p_1 + p_2 + p_3 \to p_1' + p_2' + p_3' . \tag{2.1}$$

The particles are all taken to be spinless and have equal mass. Write

$$p_i' = p_i + q_i\,, \qquad i = 1, 2, 3 \tag{2.2}$$

and

$$s_i = (p_j + p_k)^2\,,$$

$$u_1 = (p_1' + q_2)^2\,, \qquad u_2 = (p_2' + q_3)^2\,, \qquad u_3 = (p_3' + q_1)^2\,. \tag{2.3}$$

The amplitude will be regarded as a function of the nine variables $q_i^2$, $u_i$ and

$$\eta_i = u_j u_k / s_i\,. \tag{2.4}$$

In fact only eight of these are independent; there is a single non-linear relation among them [8] arising from the fact that space-time has four dimensions. This relation will be ignored in our analysis; the variables are regarded as independent and the constraint on them can be thought of as being imposed afterwards. In the asymptotic limit that we consider in this paper, this constraint involves only the variables $\eta_i$ and $q_i^2$ (see the remark after eq. (3.2) below).

We shall discuss the asymptotic limit

$$u_i \to \infty\,,$$

$$q_i^2,\ \eta_i \text{ finite}\,. \tag{2.5}$$

For suitable values of the finite variables this is actually a physical limit for the process (2.1). In this limit we find that

$$s_i'/s_i \to 1\,,$$

$$s_i' = (p' + p_k')^2\,, \tag{2.6}$$

(though the difference $s_i' - s_i$ diverges) and

$$(p_1' + q_3)^2 \sim -u_1\,, \qquad (p_2' + q_1)^2 \sim -u_2\,, \qquad (p_3' + q_2)^2 \sim -u_3\,,$$

$$(p_2 + p_3')^2 \sim (p_2' + p_3)^2 \sim -s_1 \text{ etc.} \tag{2.7}$$

In order to get an idea of the structure of $F_{\alpha_1\alpha_2\alpha_3}$, we shall suppose that the part of the amplitude that survives in the asymptotic limit (2.5) is expressible as a triple Fourier transform

$$\int_{-\infty}^{\infty} d\lambda_1\, d\lambda_2\, d\lambda_3 \int_{0}^{\infty} d\mu_1\, d\mu_2\, d\mu_3\, \psi(\lambda_i, \mu_i)\, e^{i[\Sigma\lambda_i s_i + \Sigma\mu_i u_i]}\,. \tag{2.8}$$

This is closely analogous to an assumption made in I; it must be modified if the effects of signature are to be incorporated [2]. We use eq. (2.4) to eliminate the $s_i$ and make the changes of variable

$$u_i = x_i/\mu_i,$$

$$\lambda_i = y_i\mu_j\mu_k. \tag{2.9}$$

Then the triple Mellin transform of eq. (2.8), obtained by applying the operation

$$\int_0^\infty du_1\, du_2\, du_3\, u_1^{-l_1-1}\, u_2^{-l_2-1}\, u_3^{-l_3-1} \tag{2.10}$$

to it, becomes

$$\int_{-\infty}^\infty dy_1\, dy_2\, dy_3 \int_0^\infty d\mu_1\, d\mu_2\, d\mu_3\, \mu_1^{l_1+2}\, \mu_2^{l_2+2}\, \mu_3^{l_3+2}\, \psi(y_i\mu_j\mu_k, \mu_i)$$

$$\times\, \Gamma(-l_1)\Gamma(-l_2)\Gamma(-l_3)\Omega(l_1, l_2, l_3; y_1/\eta_1, y_2/\eta_2, y_3/\eta_3)\,, \tag{2.11}$$

with

$$\Gamma(-l_1)\,\Gamma(-l_2)\,\Gamma(-l_3)\,\Omega(l_1, l_2, l_3; z_1, z_2, z_3)$$

$$= \int_0^\infty dx_1\, dx_2\, dx_3\, x_1^{-l_1-1}\, x_2^{-l_2-1}\, x_3^{-l_3-1}\, \exp\left[i\left\{\Sigma z_i x_j x_k + \Sigma x_i\right\}\right]. \tag{2.12}$$

A triple Regge pole at $l_1 = \alpha_1$, $l_2 = \alpha_2$, $l_3 = \alpha_3$ is obtained by choosing $\psi$ to have the behaviour, near $\mu_i = 0$,

$$\psi(y_i\mu_j\mu_k, \mu_i) \sim F(y_1, y_2, y_3)\, \mu_1^{-\alpha_1-3}\, \mu_2^{-\alpha_2-3}\, \mu_3^{-\alpha_3-3}. \tag{2.13}$$

From this we obtain the three-Reggeon coupling function

$$F_{\alpha_1\alpha_2\alpha_3}$$

$$= \int_{-\infty}^\infty dy_1\, dy_2\, dy_3\, F(y_1, y_2, y_3)\Omega(\alpha_1, \alpha_2, \alpha_3; y_1/\eta_1, y_2/\eta_2, y_3/\eta_3). \tag{2.14}$$

The triple-Regge-pole contribution is shown diagrammatically in fig. 1. If we make an analytic continuation in $q_1^2$ up to the point where $\alpha_1 = 0$, we obtain a bound-state pole (this comes from the factor $\Gamma(-l_1)$ in eq. (2.11)). Then fig. 1 reduces to a coupling constant times the two-Reggeon contribution to the production amplitude

$$p_2 + p_3 \to p_2' + q_1 + p_3'. \tag{2.15}$$

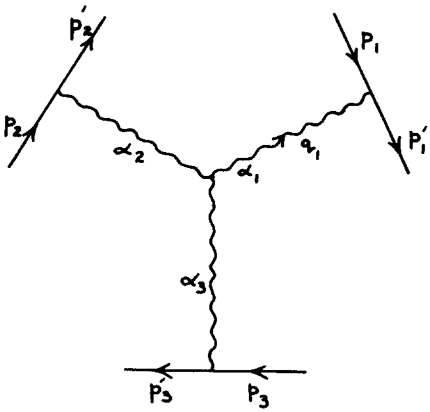


Fig. 1. The three-Reggeon contribution to the six-point function.

From eq. (2.12) we see, by integrating by parts once with respect to $x_1$, that

$$\Omega(0, l_2, l_3; z_1, z_2, z_3) = \Phi(l_2, l_3; z_1)\,, \qquad (2.16)$$

where $\Phi$ is defined in eq. (1.2). So then eq. (2.14) reduces to the form (1.1), with

$$f(y) = \int_{-\infty}^{\infty} \mathrm{d}y_2 \,\mathrm{d}y_3 \, F(y, y_2, y_3)\,,$$

and

$$\eta^{-1} = \eta_1^{-1} = (p_2 + p_3)^2 / [(p_2' + q_1)^2 (p_3' + q_1)^2]\,, \qquad (2.17)$$

which as shown in I, is linearly related to the cosine of the Toller angle for the process (2.15). That is, $F_{0\alpha_2\alpha_3}$ takes the general form of $f_{\alpha_2\alpha_3}$; in particular, it is independent of $\eta_2$ and $\eta_3$. If $p$ is a positive integer, according to eqs. (2.12) and (2.14) $F_{p\alpha_2\alpha_3}$ is a homogeneous polynomial of degree $p$ in $\eta_2^{-1}$ and $\eta_3^{-1}$.

## 3. THE GRIBOV METHOD

In this section we take, as a model for the amplitude that describes the process (2.1), the Feynman graph of fig. 2. Here the bubbles represent complete scattering amplitudes and will be supposed to have Regge-pole asymptotic behaviour.

In what follows, $ijk$ will be used to denote any cyclic permutation of 123. In the asymptotic limit (2.5) we find that

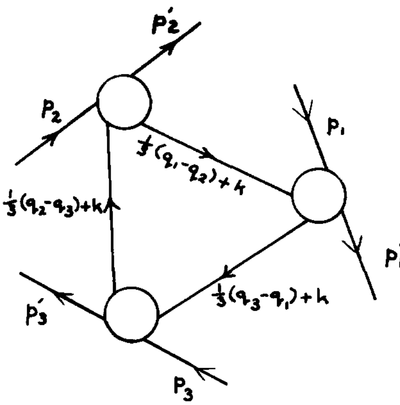


Fig. 2. Feynman graph that yields a three-Reggeon contribution. The bubbles represent complete scattering amplitudes.

$$q_i \sim p_i \left\{\frac{\eta_k - \eta_j}{2u_i} + \frac{t_k}{s_k}\right\} - p_j \left\{\frac{\eta_i(\eta_k + \eta_j)}{2\eta_j u_j} + \frac{t_i}{s_i}\right\} + p_k \left\{\frac{\eta_i(\eta_k + \eta_j)}{2\eta_k u_k}\right\} + q_i' . \tag{3.1}$$

The momenta $q_i'$ are finite and orthogonal to all three of the $p_i$. They satisfy

$$\Sigma q_i' = 0 ,$$

$$q_i'^2 = q_i^2 + \eta_i(\eta_j + \eta_k)^2/(4\eta_j\eta_k) . \tag{3.2}$$

Because the $q_i'$ are essentially one-dimensional the two relations (3.2) together yield a non-linear constraint on the variables $q_i^2$ and $\eta_i$; this was mentioned in sect. 2.

The internal momenta in the Feynman graph will be labelled as in fig. 2. We write

$$k = (z_1 p_1/u_1) + (z_2 p_2/u_2) + (z_3 p_3/u_3) + k' , \tag{3.3}$$

where $k'$ is parallel to the $q_i'$ and so is effectively a one-dimensional vector. Then, in the asymptotic limit (2.5),

$$\mathrm{d}^4 k \sim \tfrac{1}{2}(\eta_1\eta_2\eta_3)^{-\frac{1}{2}} \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}k' . \tag{3.4}$$

Following the usual spirit of the Gribov approach [7], we make this change of variables and assume that it is in order to take the limit (2.5) under the integral.

The squares of the masses on the internal lines in fig. 2 take the form

$$\sigma_i = [\tfrac{1}{3}(q_j - q_k) + k]^2$$
$$= [\tfrac{1}{3}(q'_j - q'_k) + k']^2 + (z_2 z_3/\eta_1) + (z_3 z_1/\eta_2) + (z_1 z_2/\eta_3) - \tfrac{2}{3}z_i + \tfrac{1}{3}z_j + \tfrac{1}{3}z_k$$
$$+ \tfrac{1}{9}[\tfrac{1}{2}\eta_i - \eta_j - \eta_k - (\tfrac{1}{4}\eta_i\eta_j/\eta_k) - (\tfrac{1}{4}\eta_j\eta_k/\eta_i) - (\tfrac{1}{4}\eta_i\eta_k/\eta_j)], \tag{3.5}$$

and the Reggeon energy variables are

$$[\tfrac{1}{3}(q_i - q_j) + k + p_i]^2 \sim u_i[(z_j/\eta_k) + (z_k/\eta_j) + \tfrac{1}{3}]. \tag{3.6}$$

Thus the asymptotic behaviour of the graph is

$$F_{\alpha_1\alpha_2\alpha_3} u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} [\sin(\pi\alpha_1)\sin(\pi\alpha_2)\sin(\pi\alpha_3)]^{-1} \beta_1\beta_2\beta_3, \tag{3.7}$$

where $\alpha_i = \alpha(q_i^2)$, $\beta_i = \beta(q_i^2)$, $\alpha$ and $\beta$ being respectively the Regge trajectory and the ordinary Regge residue function. Apart from some constant factors

$$F_{\alpha_1\alpha_2\alpha_3} = \int dz_1\, dz_2\, dz_3\, dk'(\eta_1\eta_2\eta_3)^{-\frac{1}{2}}$$
$$\times \prod_{i=1}^{3} \{[z_j/\eta_k + z_k/\eta_j + \tfrac{1}{3}]^{\alpha_i} g(q_i^2; \sigma_j, \sigma_k)(\sigma_i - m_i^2)^{-1}\}, \tag{3.8}$$

where $g$ is the off-shell continuation of $\beta$.

In order to show that eq. (3.8) is of the general form (2.14), we follow a procedure that begins in a way similar to that in I. First, write a triple dispersion relation (which, as discussed in I, will have no subtractions but is likely to be over a complex integration hypercontour H)

$$\prod_{i=1}^{3} \{g(q_i^2; \sigma_j, \sigma_k)(\sigma_i - m_i^2)^{-1}\} = \int_H dv_1\, dv_2\, dv_3\, \rho(q_i^2, v_i) \prod_{i=1}^{3} (\sigma_i - v_i)^{-1}. \tag{3.9}$$

Now introduce Feynman parameters $\lambda_i$:

$$(\sigma_i - v_i)^{-1} = -i \int_0^\infty d\lambda_i\, e^{i\lambda_i(\sigma_i - v_i)},$$

so that the $k'$ integration may be performed. This gives

$$F_{\alpha_1\alpha_2\alpha_3} \propto \int_0^\infty d\lambda_1\, d\lambda_2\, d\lambda_3\, \lambda^{-\frac{1}{2}} \int_H dv_1\, dv_2\, dv_3\, \rho(q_i^2, v_i) \int_{-\infty}^{\infty} dz_1\, dz_2\, dz_3$$
$$\times \prod_{i=1}^{3} [(z_j/\eta_k) + (z_k/\eta_j) + \tfrac{1}{3}]^{\alpha_i} e^{iD/\lambda} (\eta_1\eta_2\eta_3)^{-\frac{1}{2}}$$
$$\times \exp[i \sum_i \{\lambda_i \bar{\sigma}_i + \eta_i(\eta_j + \eta_k)^2 \lambda_j\lambda_k/(4\lambda\eta_j\eta_k)\}] \tag{3.10}$$

Here $\bar{\sigma}_i$ is equal to $\sigma_i$ in eq. (3.5), but with the first term absent;

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 \,,$$

and $D$ is the $D$-function [9] for the triangle graph:

$$D = \sum_i \{\lambda_j \lambda_k q_i^2 - \lambda \lambda_i v_i\} \,. \tag{3.11}$$

We now make the replacement

$$\int_{-\infty}^{\infty} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \prod [(z_j/\eta_k) + (z_k/\eta_j) + \tfrac{1}{3}]^{\alpha_i}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}Z_1 \mathrm{d}Z_2 \mathrm{d}Z_3 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$$

$$\times \exp\{i \sum_i x_i [(z_j/\eta_k) + (z_k/\eta_j) + \tfrac{1}{3} - Z_i]\} \,, \tag{3.12}$$

and perform the $z_i$ and $Z_i$ integrations. The result is

$$F_{\alpha_1 \alpha_2 \alpha_3} \propto [\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(-\alpha_3)]^{-1} \int_0^{\infty} \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3 \int \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}v_3 \, \rho$$

$$\times \int_0^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, x_1^{-\alpha_1 - 1} x_2^{-\alpha_2 - 1} x_3^{-\alpha_3 - 1} \mathrm{e}^{iD/\lambda} \lambda^{-2}$$

$$\times \exp[i\lambda^{-1} \{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 - (x_2 x_3/\eta_1) - (x_3 x_1/\eta_2) - (x_1 x_2/\eta_3)\}] \,. \tag{3.13}$$

A final simple change of integration variable brings this into the form (2.14). In particular, it can readily be checked that when $\alpha_1 = 0$, $F_{\alpha_1 \alpha_2 \alpha_3}$ becomes equal to the two-Reggeon/particle coupling extracted from a similar model in I.

## 4. THE GENERALISED VENEZIANO MODEL

In this section we investigate the three-Reggeon coupling in the generalized Veneziano model for the six-point function [5, 6], which is the simplest amplitude that can give the required coupling.

It will be convenient in this section to label the external momenta in the same way as in refs. [5, 6]. Thus fig. 1 is now replaced by fig. 3. The limit (2.5) still applies, with the scalar variables (2.3) now defined by [see also eq. (2.7)]

$$s_1 = -(p_1+p_6)^2, \quad s_2 = -(p_4+p_5)^2, \quad s_3 = -(p_2+p_3)^2,$$
$$u_1 = -(p_1+p_2+p_3)^2, \quad u_2 = -(p_2+p_3+p_4)^2, \quad u_3 = -(p_3+p_4+p_5)^2, \quad (4.1)$$

and

$$q_1^2 = (p_3+p_4)^2, \quad q_2^2 = (p_1+p_2)^2, \quad q_3^2 = (p_5+p_6)^2. \quad (4.2)$$

We use the representation for the amplitude that is given in ref. [10]:

$$\frac{1}{(2\pi i)^3}\int \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\; B(-\alpha_{56}+k_2+k_3, -\alpha_{45})$$
$$\times B(k_2+\alpha_{16}-\alpha_{345}-\alpha_{234}+\alpha_{34}, -k_2)\,B(k_3+\alpha_{345}-\alpha_{45}-\alpha_{34}, -k_3)$$
$$\times B(k_1+k_2-\alpha_{12}, -\alpha_{234})\,B(k_1+k_2+k_3+\alpha_{23}-\alpha_{234}-\alpha_{123}, -k_1)$$
$$\times B(k_1-\alpha_{34}, -\alpha_{123}+k_2+k_3)(-1)^{k_1+k_2+k_3}. \quad (4.3)$$

Here $B$ is the beta function:

$$B(\alpha,\beta) = \Gamma(\alpha)\,\Gamma(\beta)/\Gamma(\alpha+\beta)$$
$$= \int_0^{\pm\infty} \mathrm{d}x\; x^{\alpha-1}(1+x)^{-\alpha-\beta}. \quad (4.4)$$

The integrations over $k_1$, $k_2$, $k_3$ in eq. (4.4) are over contours parallel to the imaginary axis; details are given in ref. [10]. All the trajectories $\alpha$ are supposed to be linear and to have the same slope $a$. Thus in the limit (2.5) we keep finite the variables

$$\eta_1 = -\alpha_{234}\,\alpha_{345}/(a\alpha_{16}), \quad \eta_2 = -\alpha_{345}\,\alpha_{123}/(a\alpha_{45}),$$
$$\eta_3 = -\alpha_{123}\,\alpha_{234}/(a\alpha_{23}). \quad (4.5)$$

We insert the representation (4.4) for each of the beta functions in eq. (4.3) with integration variables $u$, $v$, $w$, $x$, $y$, $z$. We use the upper limit $+\infty$ for $x$, $z$, $u$ and $-\infty$ for $y$, $v$, $w$, in order to give good convergence at $|k_i| = \infty$. Then the $k_1$, $k_2$, $k_3$ integrations give us the $\delta$-functions

$$2\pi i\,\delta\left[\log\frac{-xyz}{(1+x)(1+z)}\right],$$
$$2\pi i\,\delta\left[\log\frac{-uvxy}{(1+u)(1+x)(1+y)(1+z)}\right],$$
$$2\pi i\,\delta\left[\log\frac{-uwy}{(1+u)(1+y)(1+z)}\right].$$

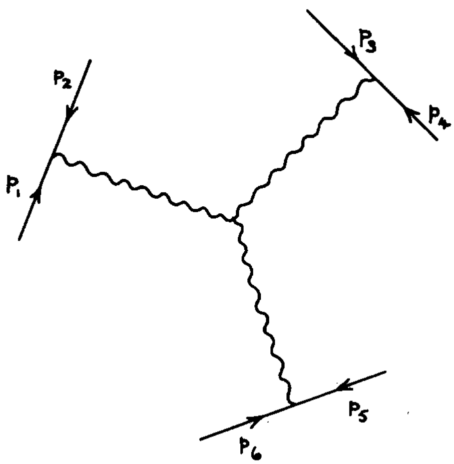


Fig. 3. The diagram of fig. 1, with the momenta relabelled as in sect. 4.

These enable us to perform the $y$, $v$, $w$ integrations:

$$y = -(1+x)(1+z)/(xz),$$

$$v = -(1+u)(1+x+z)/(xu),$$

$$w = -(1+u)(1+x+z)/[u(1+x)]. \quad (4.6)$$

Apart from a multiplying constant what remains is

$$\int_0^\infty du\,dx\,dz\; u^{-\alpha_{56}-1}(1+u)^{\alpha_{56}+\alpha_{45}}[v/(1+v)]^{\alpha_{16}-\alpha_{345}-\alpha_{234}}$$
$$\times [w/(1+w)]^{\alpha_{345}-\alpha_{45}-\alpha_{34}}\, x^{-\alpha_{12}-1}(1+x)^{\alpha_{12}+\alpha_{234}}$$
$$\times [y/(1+y)]^{\alpha_{23}-\alpha_{234}-\alpha_{123}}\, z^{-\alpha_{34}-1}(1+z)^{\alpha_{34}+\alpha_{123}}. \quad (4.7)$$

We now change to the variables

$$\bar{x} = i\alpha_{234}\, x, \qquad \bar{z} = i\alpha_{123}\, z, \qquad \bar{u} = i\alpha_{345}\, u, \quad (4.8)$$

and assume that it is in order to take the limit (2.5) inside the integration. Thus

$$\int_0^\infty dx\, x^{-\alpha_{12}-1}(1+x)^{\alpha_{12}+\alpha_{234}} \to (-i\alpha_{234})^{\alpha_{12}} \int_0^\infty d\bar{x}\, \bar{x}^{-\alpha_{12}-1}\, e^{i\bar{x}},$$

$$\int_0^\infty dz\, z^{-\alpha_{34}-1}(1+z)^{\alpha_{34}+\alpha_{123}} \to (-i\alpha_{123})^{\alpha_{34}} \int_0^\infty d\bar{z}\, \bar{z}^{-\alpha_{34}-1}\, e^{i\bar{z}},$$

$$\int_0^\infty du\, u^{-\alpha_{56}-1}(1+u)^{\alpha_{56}+\alpha_{345}} \to (-\alpha_{345})^{\alpha_{56}} \int_0^\infty d\bar{u}\, \bar{u}^{-\alpha_{56}-1}\, e^{i\bar{u}}. \quad (4.9)$$

For the remaining factors in eq. (4.7) we make the substitutions (4.6), together with eq. (4.5)

$$[v/(1+v)]^{\alpha_{16}-\alpha_{345}-\alpha_{234}} \to e^{-\bar{u}\bar{x}/a\eta_1},$$

$$[y/(1+y)]^{\alpha_{23}-\alpha_{234}-\alpha_{123}} \to e^{-\bar{x}\bar{z}/a\eta_3},$$

$$[w/(1+w)]^{-\alpha_{45}+\alpha_{345}-\alpha_{34}}(1+u)^{\alpha_{45}-\alpha_{345}} \to e^{-\bar{u}\bar{z}/a\eta_2}. \qquad (4.10)$$

When we insert eqs. (4.10) and (4.9) into eq. (4.7), we obtain an expression of the form (3.7) and (2.14), with

$$F(y_1, y_2, y_3) \propto \frac{(ia)^{\alpha_1+\alpha_2+\alpha_3}a^3}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)\Gamma(1+\alpha_3)} \frac{e^{3+ia(y_1+y_2+y_3)}}{(1+iay_1)(1+iay_2)(1+iay_3)}. \qquad (4.11)$$

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